Interim Report No. 4
Contract No. NAS8-5411

PARAMETER OPTIMIZATION

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UNIVERSITY OF ALABAMA RESEARCH INSTITUTE
Huntsville, Alabama

Interim Report No. 4
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October 15, 1964

"Parameter Optimization"

to

Director
George C. Marshall Space Flight Center
Huntsville, Alabama
Attn: M-P & C-MEA

by

UNIVERSITY OF ALABAMA RESEARCH INSTITUTE

Huntsville, Alabama

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I. INTRODUCTION

Interim Report No. 4 deals with studies on "Parameter Optimization" carried out during the period July 1, 1964 to September 30, 1964 under the contract NAS8-5411. The report contains preliminary results which should not be considered final.

In the preceding reports the effects of measurement errors on the orbital parameters in navigation and orbit determination are discussed. Relationships were derived which show the errors of the orbital parameters in terms of the errors of the observed quantities as a function of the position of the vehicle along the trajectory. By normalization, instructive relationships can be obtained which show clearly the positions along the trajectory when measurements of the radial and angular position cause excessive errors of the orbital parameters.

The present report deals with a study based on a second approach of optimization mentioned in one of the earlier progress reports. The report presents the results of a study of the application of minimum variance techniques for obtaining optimized orbital parameters. The approach is based on iterative evaluation of the measurements for the orbit determination. The values of the obtained orbital parameters come closer to those of the true orbit with each additional measurement. At this iterative approach the estimated orbit consists of a sequence of orbital trajectory segments which with each additional measurement are closer to the true orbit. Each of the trajectory segments represents a section of an optimum orbit described by optimized parameters. The optimization is valid for the specific set of the preceding measurements. Orbit perturbations can be introduced for each orbital segment and taken into account. (This study was carried out by Dr. C. H. Chambers).

One of the phases of the mathematical approach described in more detail in Part 3 (Dr. O. R. Ainsworth) is based on a generalized Newton-Raphson operator which yields in an iterative process the subsequent orbital parameters by operating on their preceding values. At these computations, it is practical to use position and velocity (previously the observed quantities) as orbital parameters. The use of the Newton-Raphson operator represents an effective method for solving nonlinear differential equations from sets of known values of the functions.

The formulations of the orbit determination described in Part 2 were converted into a computer program for the UNIVAC 1107. An example of the solution of a representative problem is shown as a conclusion of Part 2.

II. OPTIMUM ORBIT DETERMINATION BY MINIMUM VARIANCE TECHNIQUES

A. INTRODUCTION

The basic purpose of this paper is as follows. We are given a dynamic system and a procedure for making measurements on this system. We assume that if the measurements were exact, then we could predict the motion of the dynamic system exactly. However, these measurements are contaminated by noise, e.g. calibration errors, readout errors, etc. We wish to determine the time at which to make a measurement such that this measurement is optimal. A measurement is optimal if the information obtained from the measurement allows us to predict the motion of the dynamic system more accurately than any other information obtained at that or any other time. That is, the optimal measurements results in the greatest decrease in our uncertainty about the motion of the dynamic system. This approach entails two distinct optimization processes. First, we must determine the greatest decrease in uncertainty or determine the optimal information to be had at any time if a measurement were to be made at that time. Second, we must examine the presumed improvements obtained by the preceding optimizations as a function of time to determine the best time to make this measurement. This process is then repeated until the motion is known as accurately as desired.

In general the dynamic system can be described by a set of variables or coordinates x_i or equivalently a vector

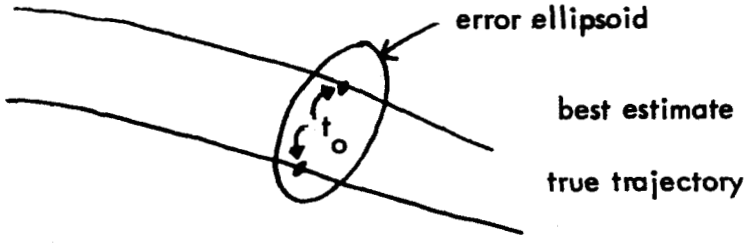
$$\underline{X} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

which obeys some differential system

$$\frac{d\underline{X}}{dt} = \underline{F}(\underline{X}, t)$$

The path of the solution $\underline{X}(t)$ in n dimensional space is called the trajectory.

If any particular measurement yielded exact values of these coordinates at some time then, the future motion could be determined exactly by solving the differential system. In reality a measurement will yield only certain coordinates with various degrees of uncertainty. Either through initial conditions or previous measurements some nominal or best estimate trajectory is known together with a set of coordinate uncertainties at each point, such as the elements of an error ellipsoid, which defines the bounds in which the system actually lies at that time.



If a measurement were made then the error bounds or error ellipsoid would be reduced in size. An optimal measurement would generally result in the greatest reduction in volume of this error ellipsoid. At each point of the nominal trajectory we will define the optimal reduction in the ellipsoid if a measurement were to be made, and then examine this set of ellipsoids along the trajectory as a function of time to determine which ellipsoid is best or equivalently at what time the measurement should be made.

B. DYNAMIC MODEL AND NEWTON-RAPHSON PERTURBATION SCHEME

a. Solution of Perturbed Dynamic Systems

We assume that the differential system can be decomposed into the form

$$\frac{d\underline{X}}{dt} = \underline{M}\ (\underline{X}, t) + \underline{P}\ (\underline{X}, t)$$

where $|\underline{M}| >> |\underline{P}|$, i.e., $\underline{P}$ is a perturbation; and the unperturbed system

$$\frac{d\underline{X}}{dt} = \underline{M}\ (\underline{X}_o, t)$$

is exactly soluble. The exact solution can be characterized by some set of system elements $\lambda = (\lambda_1, \ldots, \lambda_n)^T$. One possible set of elements is the system vector at a particular time, $\underline{X}\ (t_o)$.

Initially, the unperturbed solution, $\underline{X}_o$, will be a close approximation to the true solution. However, since $\underline{P}\ (t) \neq 0$ the system will eventually deviate significantly from the unperturbed solution and $\underline{X}_o$ will no longer be a good approximation. This difficulty can be corrected by the process of discrete variation of elements, i.e., rectification. When the system deviates from the unperturbed solution by predetermined limits, the system elements are re-adjusted so that the deviation from a new unperturbed solution, $\underline{X}_o$ is zero.

One standard method of integrating the above differential system is to let $\underline{X} = \underline{X}_o + \underline{\xi}$, and then to numerically integrate the resulting nonlinear perturbation equation,

$$\frac{d\underline{\xi}}{dt} = (\underline{M}\ (\underline{X}_o + \underline{\xi}, t) - \underline{M}\ (\underline{X}_o, t)) + \underline{P}\ (\underline{X}_o + \underline{\xi}, t).$$

The usual difficulties encountered in numerically integrating nonlinear differential

equations are reduced because both $\underline{\xi}$ and its derivative are small. If $\underline{\xi}$ becomes too large, the rectification process described above is used. This is the well known Encke method. However, this method makes use of the exact solution only to the extent of formulating the perturbation differential equations.

We propose a completely different approach to this problem which makes much more analytic use of the exact solution. We shall construct a sequence of solutions, $\underline{X}^m$ which converge to the desired solution in the Cauchy sense, i.e., $\|\underline{X}^n - \underline{X}^m\| < \quad n, m > N$. We take the first element in the sequence $\underline{X}^o$ to be $\underline{X}_o$. Successive elements are obtained by operating on preceding elements with the generalized Newton-Raphson operator, i.e., $\underline{X}^{n+1} = (NR)\,\underline{X}^m$. The Newton-Raphson operator is defined by the solution of an associated linear differential system.

$$\frac{d\underline{X}^{m+1}}{dt} = \underline{J}(\underline{X}^m)(\underline{X}^{m+1} - \underline{X}^m) + \underline{F}(\underline{X}^m, t)$$

$$\underline{F}(\underline{X}, t) = \underline{M}(\underline{X}, t) + \underline{P}(\underline{X}, t), \quad \underline{J} = \frac{\partial(\underline{F})}{\partial(\underline{X})}, \text{ the Jacobian}$$

This operator can be shown to be a contractive mapping of the metric space of the variables into itself, thus only one point (solution) is invariant and the iterative scheme will converge. Also, the fact that the iterates are obtained from a linear differential system, allows a constant, relatively large step length to be used in their numerical integration. This in contrast to the numerical necessity of variable step length when forwardly integrating a nonlinear system.

This result at first may seem paradoxical since for large n the solution of the associated linear system is effectively the same as the exact solution of the nonlinear system, and the numerical difficulties in obtaining these solutions should be the same. However, it must be recalled that any forward integration scheme involves a sampling of the family of solutions about the desired solution,

and the degree to which the neighboring families approximate the desired solution determines the accuracy of the scheme with a given step length. Thus even though the solutions of the linear and nonlinear systems are the same, the associated families will be quite different and the family for the linear system will be much more well behaved.

The effectiveness of any iterative scheme depends on the first element of the sequence. Experience has shown that for the problem under consideration, three iteration are sufficient, using the exact solution $\underline{X}_o$ as the first element.

b. Illustrative Example

As an example which will demonstrate the Newton-Raphson technique, consider the nonlinear equation

$$\frac{dy}{dt} = \epsilon y^2 \qquad |\epsilon| << 1 \qquad y(0) = 1$$

Clearly, the solution of the unperturbed part ($\epsilon = 0$) is

$$y_o = 1$$

The jacobian is

$$J = 2\epsilon y \ ,$$

and the first iterate is determined from the linear equation

$$\frac{dy^1}{dt} = 2\epsilon (y^1 - 1) + \epsilon$$

$$= 2\epsilon y - \epsilon$$

This equation has the solution

$$y^1 = \frac{1}{2} (1 + e^{2\epsilon t})$$

Comparing this to the exact solution

$$y = \frac{1}{1 - \epsilon t} \quad ,$$

We see that they agree to the third order in ϵ from just one iteration.

c. Satellite Orbits and On-Board Modification of Perturbation Scheme

Let us now examine the actual dynamic system under consideration, i.e., satellite orbits. For satellites of the earth, the exactly solvable part is just the classical 2-body problem. The perturbation term may contain as many effects as the investigation desires. However, for close-in orbits (within 3 earth radii), the major perturbation is the oblatness of the earth. Since the derivations and computations presented here are of a pilot nature, we will limit ourselves to include different perturbations.

We take our dynamic coordinate to be a quasi-inertial rectangular, cartesian coordinate system, centered at the center of the earth. The z axis is aligned in the right hand sense with the earth's axis of rotation; the α axis points toward the vernal equinox in Aries; the y-direction completes the right-hand triad. The system is quasi-inertial because its acceleration, i.e., the acceleration of the earth toward the sun, is a neglected perturbation.

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1}^{3} p_i^2 = \frac{K}{r} \left[1 + \frac{2}{3} J \left(\frac{R}{r}\right)^2 P_2 (\cos \theta) \right]$$

K is the gravational constant, $R^2 = \sum_{i=1}^{3} X_i^2$, R is the radius of the earth, and J is the shape factor. We have truncated the Legendre expansion after the oblatness term. We have also set the P_1 coefficient equal to zero, since the center of the earth effectively coincides with the center of the coordinate system.

The Hamiltonian equations of motion are

$$\dot{X}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$\dot{p}_i = - \frac{\partial \mathcal{H}}{\partial x_i}$$

Making the standard replacement of p_i by $m X_{i+3}$, we have the differential system

$$\dot{X}_i = X_{i+3}$$

$$\dot{X}_{i+3} = - \frac{KX_i}{r_3} \left[1 + J \left(\frac{R}{r}\right)^2 \underline{P}_{2i} \right] \qquad i = 1, 2, 3$$

where

$$\underline{P}_{21} = \underline{P}_{22} = 1 - 5 \frac{X_3^2}{r^2} = \underline{P}_{23} - 2$$

are the partial derivatives of the Legendre polynomial term.

In order to apply the Newton-Raphson scheme, we need to determine the jacobian. From the above form of the forcing matrix, we have

$$\frac{\partial (\underline{F})}{\partial (\underline{X})} = \begin{pmatrix} 0 & I \\ H & 0 \end{pmatrix}$$

where the sub matrices are 3 x 3 and the components of H are obtained by straight forward differentiation of the forcing matrix. We have

$$h_{ij} = \frac{K (3 x_i x_j - \delta_{ij} r^2)}{r^5} \left[1 + J \left(\frac{R}{r}\right)^2 P_{2i} \right]$$

$$+\frac{2KJR^2X_i}{r^7}\left(X_i P_{2i} - \frac{5X_3}{r^2}\left(X_3 X_i = \delta_{3i} r^2\right)\right)$$

In order to begin the Newton-Raphson iterations, it is necessary to obtain the exact two-body solution. The most practical method of calculating this solution from the point of view of the rectification process and elliptic orbits is to use the position and velocity vectors at an initial time as basis vectors, and expand the position and velocity vectors at a later time in this basis. This is clearly possible since the two-body orbit lies in a fixed plane, and any non-colinear position and velocity vectors are sufficient to define this plane. We have the well documented results.

$$\vec{r} = P\vec{r}_o + Q\dot{\vec{r}}_o$$

$$\dot{\vec{r}} = \dot{P}\vec{r}_o + \dot{Q}\dot{\vec{r}}_o$$

where

$$a = \left(\frac{2}{r} - \frac{V^2}{K}\right)^{-1} \qquad d_o = \vec{r}_o \cdot \dot{\vec{r}}_o$$

$$P = -\frac{a}{r_i}\left(1 - \cos\theta\right) + 1$$

$$Q = \sqrt{\frac{a}{K}}\, r_o \sin\theta + \frac{ad_o}{K}\left(1 - \cos\theta\right)$$

$$r = a\left(1 - \cos\theta\right) + r_o \cos\theta + \sqrt{\frac{a}{K}}\, d_o \sin\theta$$

$$\dot{P} = -\sqrt{\frac{ak}{rr_o}} \sin\theta$$

$$\dot{Q} = \frac{a}{r}\left(1 - \cos\theta\right) + 1$$

θ , the increment in eccentric anomaly is determined from the generalized Kepler's equation

$$\mu (t - t_o) = \theta - \sin\theta + \frac{r_o}{a} \sin\theta + \frac{d_o}{\sqrt{aK}} (1 - \cos\theta)$$

$$\mu = \sqrt{\frac{K}{a^3}}$$

This trancendental equation is solved by Newton's method of iteration.

In the Encke method rectification is done whenever a component of the perturbation vector $\underline{\xi}$ exceeds a predetermined limit. However, in the Newton-Raphson scheme, the criterion for accuracy is how "close" the last two iterates lie to each other. The measure of "closeness" is conveniently defined as the metric of a metric space. A computationally attractive, yet analytically meaningful, metric useful for our purposes is given by

$$\rho (\underline{X}^n , \underline{X}^m) = \sum_{i=1}^{6} \max_{t \in [t_o, t_f]} | X_i^n (t) - X_i^m (t) |$$

When this scalar quantity, taken between the last two iterates, exceeds a specified limit, the perturbation scheme is considered to be diverging, and rectification is applied. Clearly, the accuracy could be improved by calculating more iterates. However, in order to perform these additional calculations, a greatly expanded computer memory capacity is required. Also, much more time is used than in the rectification process. By fixing the maximum number of iterates ahead of time, a great savings in computer memory storage can be had. This savings makes the scheme attractive for use in limited size, on board computers. Let us now examine how this savings is effected.

When one numerically calculates the Newton-Raphson solution, two distinct iteration processes are involved. First, the solution iterates are obtained one from the other; and second, the independent variable, time, is iterated over a fixed step length in numerically integrating the associated linear system.

If time is iterated first, the entire solution iterate must be remembered in order to obtain the next solution iterate. This requires a great deal of computer storage. However, if the solutions are iterated first, and then the time, almost no memory is required since the integration scheme, e.g. Runge-Rutta, requires only a limited number of neighboring values of the solution iterates at each integration point. The only restriction of the method is that the maximum number of solution iterates are fixed initially.

d . Derivation of the Optimization Parameter

Let us now examine the mathematical specifics of the general optimization scheme discussed in the introduction.

Whenever a measurement is made, we obtain an associated set measurement errors. We assume that these errors are known experimentally for the type of measuring apparatus used. It is also possible for these measurements errors to depend on the values of the measurements themselves. These errors represent the accuracy to which we are able to determine the state of our system. These errors can be represented by a matrix $\mathcal{Y}$, in which the diagonal elements are the standard deviations squared, and the off diagonal elements are the correlations.

Concurrently, the minimum variance calculations yields the matrix Y^+ . The elements of Y^+ are the elements of the predicted error ellipsoid in measurement space. This ellipsoid measures the improvement in uncertainty about the dynamic system, if the measurement were to be made.

We can determine how optimal the results of our measurement are by just comparing Y^+ and $\mathcal{Y}$. The degree to which Y^+ ellipsoid duplicates the $\mathcal{Y}$ ellipsoid in both size, shape, and orientation determines how optimal the measurement is. Let us now examine these two criteria.

First, assume that Y^+ and $\mathcal{Y}$ represent ellipsoids of the same shape, and similarly oriented. Consider the ratio of the volume of Y^+ to that of $\mathcal{Y}$. If two ratios for two measurements are larger than unity, then the one closer to unity is more optimal. If the ratio is less than one, we have an experimentally unverifiable situation. However, just from considerations of the system errors, the ratio closer to zero is more optimal.

Second, assume that the two ellipsoids have the same volume. Then, of two measurements, the one more nearly proportional (the same shape and orientation) to $\mathcal{Y}$ is more optimal.

Now, the square roots of the diagonal elements of Y^+ and $\mathcal{Y}$ are the standard deviations and also the half dimensions of the axis oriented parallelipiped

just containing the respective ellipsoids. If we use these bounding parallelipipeds for comparison, we effectively ignore the amount the ellipsoid is rotated from the coordinate axes. These angles are just a measure of the correlations between various components in measurement space, and ignoring them does not affect our knowledge about the absolute errors.

Let us define

$$\sigma_{oi} = \sqrt{|\Upsilon_{ii}|}$$

$$\sigma_i = \sqrt{|\Upsilon^{+}_{ii}|} \qquad \text{(not-summed)}$$

$$\bar{\sigma}_o = \frac{1}{n} \sum_{i=1}^{n} \sigma_{oi}$$

$$\bar{\sigma} = \frac{1}{n} \sum_{i=1}^{n} \sigma_i \qquad n = \text{dimensions of observation space.}$$

Then an absolute measure of the shape of the parallelipiped is given by the quantities

$$\gamma_{oi} = \sigma_{oi} / \bar{\sigma}_o$$

$$\gamma_i = \sigma_i / \bar{\sigma}$$

and a measure of the volume is clearly

$$V_o = (\sigma_o)^n$$

$$V = (\bar{\sigma})^n \quad .$$

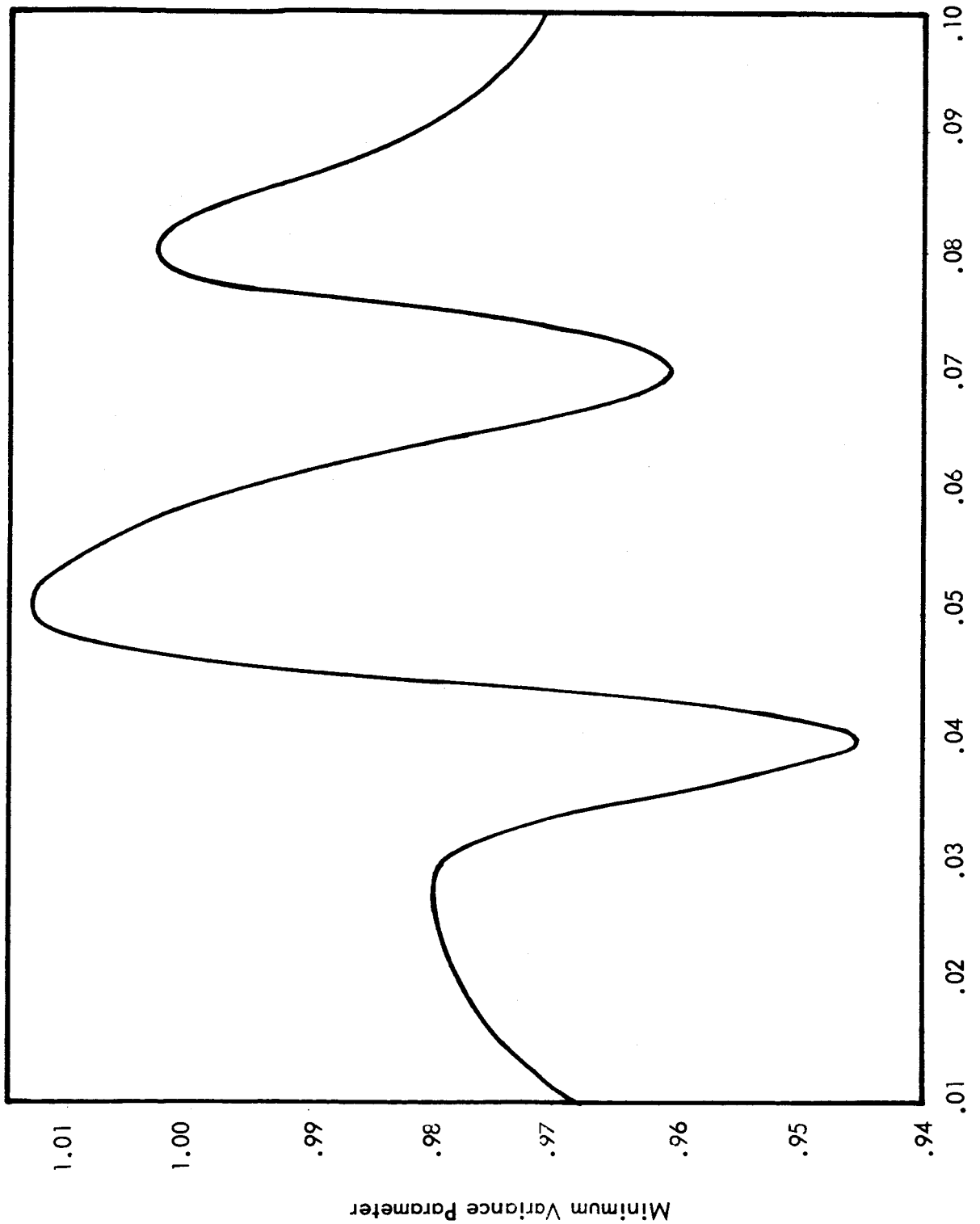

Minimum Variance Parameter
1.01
1.00
.99
.98
.97
.96
.95
.94
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
Relative Time

Figure 1

A measure of the deviations in shape is given by

$$\alpha = \sum_{i=1}^{n} (\gamma_i - \gamma_{oi})^2$$

and in volume, by

$$V'/V_o = \left(\frac{\bar{\sigma}}{\bar{\sigma}_o}\right)^n$$

To accent the shape deviations we write e^{α} and since we used the squares to obtain absolute magnitudes we take the square root, i.e.

$$e^{\frac{1}{2}\alpha}$$

Since we want both shape and size effects, we merely multiply them to obtain the optimization parameter ϕ as,

$$\phi = \left(\frac{\bar{\sigma}}{\bar{\sigma}_o}\right)^2 e^{\frac{1}{2}\sum_{i=1}^{n} (\gamma_i - \gamma_{io})^2}$$

For a typical satellite orbit the variation of ϕ with time is shown in Figure 1.

C. DIFFERENTIAL CORRECTION AND PARAMETER OPTIMIZATION PROGRAM USING MINIMUM VARIANCE TECHNIQUE

a. Probability Theory and Classical Minimum Variance Estimates

Let us consider a set of n random real variables $y(t_1), \cdots, y(t_n)$. The probability that all the variables are less than a specified set of values is called the joint probability distribution function

$$P_r [y(t_1) \leq z_1, \cdots, y(t_n) \leq z_n]$$

$$= F(y(t_1), \cdots, y(t_n); z_1, \cdots, z_n)$$

The joint probability density function is given by

$$f(y(t_1), \cdots, y(t_n); z_1, \cdots, z_n)$$

$$= \frac{\partial^n F(y(t_1), \cdots, y(t_n); z_1, \cdots, z_n)}{\partial t_1 \cdots \partial t_n}$$

The expectation of a function $g(y(t_1), \cdots, y(t_n))$ is defined to be

$$E[g(y(t_1), \cdots, y(t_n))] =$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y(z_1), \cdots, y(z_n)) f(y(t_1), \cdots, y(t_n); z_1, \cdots, z_n) dz_1 \cdots dz_n$$

The conditional expectation is given by

$$E[g(y(t_n)) \mid y(t_1), \cdots, y(t_{n-1})]$$

$$= \int_{-\infty}^{\infty} \frac{g\,(y\,(z_n))\, f\,(y\,(t_1), \cdots, y\,(t_n)\,;\, z_1, \cdots, z_n)\, dz_n}{f\,(y\,(t_1), \cdots, y\,(t_{n-1})\,;\, z_1, \cdots, z_{n-1})}$$

where the $y\,(t_1), \ldots, y\,(t_{n-1})$ are specified. We have by the above definitions

$$E\,[\,E\,[g\,(y\,(t_n))\;\big|\;y\,(t_1), \cdots, y\,(t_{n-1})\,]\,]$$

$$= E\,[\,g\,(y\,(t_n))\,]$$

Let us now examine the specific details of the prediction problem. Let us have a distinct set of time ordered observational data given:

$$\underline{y}\,(t_1), \cdots, \underline{y}\,(t_n)$$

$$t_i > t_j \text{ iff } i > j$$

$$\underline{y}\,(t_i) = \begin{vmatrix} y_1\,(t_i) \\ \vdots \\ y_p\,(t_i) \end{vmatrix}$$

The $y_i(t_j)$ are real-valued random variables.

Using all of this information we can predict through a system of regression relations some optimal estimate $\hat{\underline{X}}\,(t)$ of the trajectory. This trajectory will deviate from the true $\underline{X}\,(t)$ by some amount

$$\underline{\varepsilon} = \underline{X}\,(t) - \hat{\underline{X}}\,(t).$$

$\underline{\varepsilon}$ is a measure of our incorrect prediction. To obtain the optimal prediction, we should minimize the expectation value of some positive definite, monotonically increasing function of $\underline{\varepsilon}$ with respect to the prediction. This function is

appropriately called the loss or penalty function. The usual choice for this function is $\underline{\varepsilon}\ \underline{\varepsilon}^T$ where $\underline{\varepsilon}^T$ is the transpose of $\underline{\varepsilon}$. This choice can be justified or unjustified depending on the source. We will use it because of its mathematical attractiveness, and leave it at that. Notice that $\underline{\varepsilon}$ is a column vector and that $\underline{\varepsilon}\ \underline{\varepsilon}^T$ thus has no meaning as a matrix product. This is defined to be the direct or dyadic product of the two vectors. We have from above

$$E\left[\underline{\varepsilon}\ \underline{\varepsilon}^T\right] = E\left[E\left[\underline{\varepsilon}\ \underline{\varepsilon}^T \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right]\right]$$

and to obtain the minimum of $E\left[\underline{\varepsilon}\ \underline{\varepsilon}^T\right]$ we need only minimize the conditional expectation.

$$E\left[\underline{\varepsilon}\ \underline{\varepsilon}^T \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right]$$

Proceeding formally we have

$$\underline{\varepsilon}\ \underline{\varepsilon}^T = (\underline{X} - \hat{\underline{X}})\ (\underline{X} - \hat{\underline{X}})^T$$

$$\frac{\partial}{\partial \hat{\underline{X}}} \left\{ E\left[\underline{\varepsilon}\ \underline{\varepsilon}^T \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right]\right\} = 0$$

$$= \frac{\partial}{\partial \hat{\underline{X}}} \left\{ E\left[\underline{X}\,\underline{X}^T \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right]\right.$$

$$-2\,\hat{\underline{X}}\, E\left[\underline{X}^T \,\middle|\, \underline{y}(t_1), \cdots, y(t_n)\right]$$

$$\left. + \hat{\underline{X}}\,\hat{\underline{X}}^T\, E\left[1 \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right]\right\}$$

$$= -2E\left[\underline{X} \,\middle|\, \underline{y}(t_1), \cdots, \underline{y}(t_n)\right] + 2\hat{\underline{X}} = 0$$

where we have use the formal relations

$$\frac{\partial \hat{\underline{X}} \underline{X}^T}{\partial \hat{\underline{X}}} = \frac{\partial \underline{X} \hat{\underline{X}}^T}{\partial \hat{\underline{X}}} = \underline{X} ,$$

$$\frac{\partial \hat{\underline{X}} \hat{\underline{X}}^T}{\partial \hat{\underline{X}}} = 2 \hat{\underline{X}}$$

and the fact that

$$E [1 \mid \underline{y}(t_1), \cdots, \underline{y}(t_n)] = 1 .$$

This yields the optimal estimate

$$\hat{\underline{X}} = E [\underline{X} \mid \underline{y}(t_1), \cdots, \underline{y}(t_n)]$$

Notice that this approach tacitly assumes that all the observations are known. Since $E [\underline{\varepsilon} \underline{\varepsilon}^T]$ is the variance, this method is frequently called the minimum variance technique. These results can be expressed in a much more elegant formalism. This formalism will also enable us to attack the differential correction problem in a straightforward, concise manner.

b. Minimum Variance Estimates in Observation Space, and Wiener Differential Correction Problem

Any random variable which can be written in the form

$$\sum_{i,j} A_{ij} y_i (t_j)$$

can be considered a vector in the vector space with basis the independent random variables $y_i(t_j)$. Clearly, the A_{ij} are the components of the given variable. Denote the vector space generated in this manner by $\underline{y}(t_1), \cdots, \underline{y}(t_n)$ as $Y(t_n)$, the observation space. (Recall that we have required our observations to be time

ordered. Thus specifying the end time t_n is sufficient to specify the entire sequence). The scalar product between two vectors is just their covariance, i.e., $\underline{x} \cdot \underline{y} = E(\underline{x}^T \underline{y})$. The usual definitions of orthogonality, length, orthonormal basis, etc., follow directly.

Any random variable, $\underline{x}(t)$, can be decomposed (uniquely) into a part $\underline{x}_{11}(t \mid t_n)$ lying in $Y(t_n)$, and a part $\underline{x}_{\perp}(t \mid t_n)$ perpendicular or orthogonal to $Y(t_n)$. Now if the random processes are gaussian, then the orthogonal projection $\underline{x}_{11}(t \mid t_n)$ is just the conditional expectation of $\underline{x}(t)$ with respect to $Y(t_n)$ i.e.,

$$\underline{x}_{11}(t \mid t_n) = E[\underline{x}(t) \mid \underline{y}(t_1), \cdots, \underline{y}(t_n)]$$

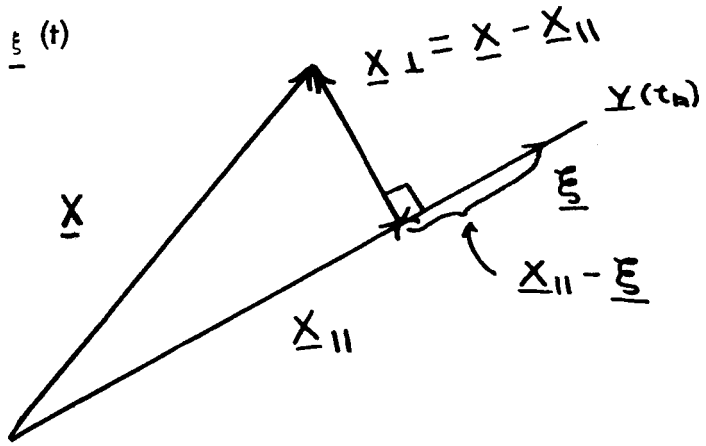
$$\equiv E[\underline{x}(t) \mid Y(t_n)]$$

But this conditional expectation is just what we obtained above for the optimal estimate, $\hat{\underline{X}}(t)$, of $\underline{X}(t)$. Thus we are led to propose:

$$\hat{\underline{X}}(t \mid t_n) = \underline{X}_{11}(t \mid t_n) = E[\underline{X}(t) \mid Y(t_n)]$$

and we wish to prove that it is unique. Proof: Let $\underline{\xi}(t)$ be any estimate of $\underline{X}(t)$ in $Y(t_n)$. We have

$$\underline{\varepsilon} = \underline{X}(t) - \underline{\xi}(t)$$



Variance $(\underline{\varepsilon}) = E[\underline{\varepsilon}\,\underline{\varepsilon}^T]$

$$= E[(\underline{X} - \underline{\xi})(\underline{X} - \underline{\xi})^T]$$

$$= E\left[\left((\underline{X} - \underline{X}_{11}) + (\underline{X}_{11} - \underline{\xi}) \right) \left((\underline{X} - \underline{X}_{11}) + (\underline{X}_{11} - \underline{\xi}) \right) \right]$$

Since $(\underline{X} - \underline{X}_{11})$ is orthogonal to $(\underline{X}_{11} - \underline{\xi})$ we have

$$E(\underline{\varepsilon}\,\underline{\varepsilon}^T) = E\left[(\underline{X} - \underline{X}_{11})(\underline{X} - \underline{X}_{11}) \right] + E\left[(\underline{X}_{11} - \underline{\xi})(\underline{X}_{11} - \underline{\xi}) \right]$$

$\geq E\left[(\underline{X} - \underline{X}_{11})(\underline{X} - \underline{X}_{11}) \right]$, the minimum variance, unless $\underline{\xi} = \underline{X}_{11}$ Q.E.D. This is clearly equivalent to saying that the projection that minimizes the distance (here variance) from a vector to a subspace is just the orthogonal projection. The optimal estimate of the state vector $\underline{X}(t_k)$ for measurements made through that time is clearly

$$\hat{\underline{X}}(t_k \mid t_k) = E\left[\underline{X}(t_k) \mid Y(t_k) \right]$$

We will use this relation to generate a recurrence relation for $\hat{\underline{X}}$. Now, the measurements through t_{k-1} generate the vector space $Y(t_{k-1})$, and the measurement $\underline{y}(t_k)$ adds to this the subspace $Z(t_k)$, i.e.

$$Y(t_k) = Z(t_k) + Y(t_{k-1})$$

Every vector in $Z(t_k)$ is orthogonal to every vector in $Y(t_{k-1})$. $\underline{y}(t_k)$ can be decomposed as follows

$$\underline{y}(t_k) = \underline{y}_{\perp}(t_k \mid t_{k-1}) + \underline{y}_{11}(t_k \mid t_{k-1}),\ \underline{y}_{\perp}(t_k \mid t_{k-1})$$

$\in Z(t_k)$, $\underline{y}_{11}(t_k \mid t_{k-1}) \in Y(t_{k-1})$ and $\underline{X}(t_k)$ can be obtained from $\underline{X}(t_{k-1})$ by the solution of the original differential system, i.e.

$$\underline{X}(t_k) = S\left\{ \underline{X}(t_{k-1}), t_k \right\} .$$

Thus we have

$$\hat{\underline{X}}(t_k \mid t_k) = E\left[\underline{X}(t_k) \mid Z(t_k) + Y(t_{k-1}) \right]$$

$$= E \left[S \left\{ \underline{X} (t_{k-1}), t_k \right\} \;\middle|\; Y (t_{k-1}) \right]$$

$$+ E \left[\underline{X} (t_k) \;\middle|\; Z (t_k) \right]$$

The first term is just the dynamic prediction of $\hat{\underline{X}} (t_k)$ from the best information had at time t_{k-1}, i.e., $Y (t_{k-1})$. The second term represents the additional information gained from the measurement $\underline{y} (t_k)$. If no new information is gained from $\underline{y} (t_k)$, then $\underline{y} (t_k) \in Y (t_{k-1})$, the subspace $Z (t_k)$ is zero, the second term is zero, and the optimal estimate is just the dynamic prediction of the previous optimal estimate. Clearly this is a perfectly consistent result. The additional information must be due only to that part of $\underline{y} (t_k)$ lying in $Z (t_k)$, i.e., $\underline{y}_\perp (t_k \mid t_{k-1})$. Thus we must have

$$E \left[\underline{X} (t_k) \;\middle|\; Z (t_k) \right] = W \left(\underline{y}_\perp (t_k \mid t_{k-1}) \right)$$

where $W (\underline{y})$ is the Wiener optimal estimator function. If the stochastic processes are gaussian, then the optimal estimator is a linear function of its argument. In this case W becomes a matrix and we have

$$W \left(\underline{y}_\perp (t_k \mid t_{k-1}) \right) = W (t_k) \; \underline{Y}_\perp (t_k \mid t_{k-1})$$

Only if the processes are non-gaussian (thus requiring third order statistics) can our estimation be improved by using a non-linear function for $W (\underline{y})$. Thus we have

$$\hat{\underline{X}} (t_k \mid t_k) = E \left[S \left\{ \underline{X} (t_{k-1}), t_k \right\} \;\middle|\; Y (t_{k-1}) \right] + W \underline{y}_\perp (t_k \mid t_{k-1})$$

Now

$$\underline{y} (t_k) = D \left\{ \underline{X} (t_k) \right\} + \underline{n}$$

where $\underline{n}$ is the noise and errors vector and

$$\underline{y}_{11} (t_k \mid t_{k-1}) = D \left\{ \hat{\underline{X}} (t_k \mid t_{k-1}) \right\} + \underline{n}$$

since $\underline{y}_{11}(t_k \mid t_{k-1}) \in Y(t_{k-1})$ and $\underline{X}(t_k \mid t_{k-1})$ is the projection of $\underline{X}(t_k)$ on $Y(t_{k-1})$. Assuming the expansion

$$D(\underline{X}') - D(\underline{X}'') = G(\underline{X}')(\underline{X}' - \underline{X}'') + 0(X' - X'')^2 ,$$

we have

$$\underline{y}_{\perp}(t_k \mid t_{k-1}) = G(t_k)\,(\underline{X}(t_k) - \hat{\underline{X}}(t_k \mid t_{k-1})) + \underline{n}$$

$$= G(t_k)\,\Delta\underline{X}(t_k \mid t_{k-1}) + \underline{n}$$

where

$$\Delta\underline{X}(t_k \mid t_{k-1}) = \underline{X}(t_k) - \hat{\underline{X}}(t_k \mid t_{k-1})$$

Thus we have

$$\Delta\underline{X}(t_k \mid t_k) = \underline{X}(t_k) - \hat{\underline{X}}(t_k \mid t_k)$$

$$= S\{\underline{X}(t_{k-1}), t_k\} - E[S\{\underline{X}(t_{k-1}), t_k\} \mid Y(t_{k-1})]$$

$$- WG\,\Delta\underline{X}(t_k \mid t_{k-1}) .$$

where $W\underline{n}$, the optimal estimation of noise, is zero. Assuming the expansion

$$S\{\underline{X}'(t_1), t_2\} - S\{\underline{X}''(t_1), t_2\}$$

$$= \Phi(t_2, t_1)(\underline{X}'(t_1) - \underline{X}''(t_1)) + O(\underline{X}'(t_1) - \underline{X}''(t_1))^2 ,$$

where Φ is the state transition matrix. We have

$$\Delta\underline{X}(t_k \mid t_k) = \Phi(t_k, t_{k-1})\;\Delta\underline{X}(t_{k-1} \mid t_{k-1})$$

$$- WG \Delta \underline{X} (t_k \mid t_{k-1}) \quad .$$

Since

$$E [\Phi (t_k, t_{k-1}) \underline{X} (t_{k-1}) \mid Y (t_{k-1})] = \Phi(t_k, t_{k-1}) E [\underline{X} (t_{k-1}) \mid Y (t_{k-1})]$$

$$= \Phi (t_k, t_{k-1}) \hat{\underline{X}} (t_{k-1} \mid t_{k-1})$$

Now

$$\Delta \underline{X} (t_k \mid t_{k-1}) = \underline{X} (t_k) - \hat{\underline{X}} (t_k \mid t_{k-1})$$

$$= S \{ \underline{X} (t_{k-1}), t_k \} - E [S \{ \underline{X} (t_{k-1}), t_k \} \mid Y (t_{k-1})]$$

$$= S \{ \underline{X} (t_{k-1}), t_k \} - S \{ E [\underline{X} (t_{k-1}) \mid Y (t_{k-1})] , t_k \}$$

$$= S \{ \underline{X} (t_{k-1}), t_k \} - S \{ \hat{\underline{X}} (t_{k-1} \mid t_{k-1}), t_k \}$$

$$= \Phi (t_k, t_{k-1}) \quad \Delta \underline{X} (t_{k-1} \mid t_{k-1})$$

Thus

$$\Delta \underline{X} (t_k \mid t_k) =$$

$$= \Phi (t_k, t_{k-1}) \quad \Delta \underline{X} (t_{k-1} \mid t_{k-1}) = (I - WG) \quad \Delta \underline{X} (t_k \mid t_{k-1})$$

$$- WG \Phi (t_k, t_{k-1}) \Delta \underline{X} (t_{k-1} \mid t_{k-1}) \quad .$$

Let us denote the covariance matrix of $\Delta \underline{X} (t_{k+1} \mid t_k)$ by $\pi (t_{k+1})$. We have

$$\pi (t_{k+1}) = E [\Delta \underline{X} (t_{k+1} \mid t_k) \Delta \underline{X}^T (t_{k+1} \mid t_k)]$$

$$= E [\Phi (t_{k+1}, t_k) \Delta \underline{X} (t_k \mid t_k) \Delta \underline{X}^T (t_k \mid t_k) \Phi^T (t_{k+1}, t_k)]$$

$$= E\,[\Phi(t_{k+1}, t_k)\,[I - WG]\,\Delta\underline{X}(t_k \mid t_{k-1})\,\Delta\underline{X}^T(t_k \mid t_{k-1})\,[I - WG]^T\Phi^T(t_{k+1}, t_k)]$$

$$= \Phi(t_{k+1}, t_k)\,\pi(t_k)\,\Phi^T(t_{k+1}, t_k) - \Phi(t_{k+1}, t_k)\,WG\,\pi(t_k)\,\Phi^T(t_{k+1}, t_k)$$

where we have used the fact that

$$[WG\,\Delta\underline{X}(t_k \mid t_{k-1})] = [W\underline{y}_\perp(t_k \mid t_{k-1})]$$

$$= E\,[\underline{X}(t_k) \mid Z(t_k)] \quad \text{is a random variable.}$$

Thus we have

$$\pi(t_{k+1}) = \Phi(t_{k+1}, t_k)\,[I - WG]\,\pi(t_k)\,\Phi^T(t_{k+1}, t_k)\,.$$

If no additional information were obtained at t_{k+1} then W would be zero and $\pi(t)$ would be given by $\pi = \Phi\,\pi\,\Phi^T$, a reasonable result if Φ is considered to be a state transition matrix.

We must now determine the optimal estimator W. We have by definition

$$E\,[\underline{X}(t_k) \mid Z(t_k)] = W(t_k)\,\underline{y}(t_k \mid t_{k-1})$$

i.e. the component of $\underline{X}(t_k)$ lying in $Z(t_k)$ is $W\underline{y}_\perp(t_k \mid t_{k-1})$.

Thus

$$E\,[\underline{X}(t_k)\,\underline{Y}_\perp^T(t_k \mid t_{k-1})]$$

$$= E\,[W(t_k)\,\underline{y}_\perp(t_k \mid t_{k-1})\,\underline{y}_\perp^T(t_k \mid t_{k-1})]$$

$$= W(t_k)\,E\,[(G(t_k)\,\Delta\underline{X}(t_k \mid t_{k-1}) + \underline{n})\,(G(t_k)\,\Delta\underline{X}(t_k \mid t_{k-1}) + \underline{n})^T]$$

$$= W(t_k)\,[G(t_k)\,\pi(t_k)\,G^T(t_k) + N]$$

where $N = E [\underline{n} \underline{n}^T]$.

Now $E [\hat{\underline{X}} (t_k \mid t_{k-1}) \underline{y}_\perp^T (t_k \mid t_{k-1})] = 0$

since $\hat{\underline{X}} (t_k \mid t_{k-1}) \in Y (t_{k-1})$

thus $E [\underline{X} (t_k) \underline{y}_\perp^T (t_k \mid t_{k-1})]$

$= E [(\underline{X} (t_k) - \hat{\underline{X}} (t_k \mid t_{k-1})) \underline{y}_\perp^T (t_k \mid t_{k-1})]$

$= E [\Delta \underline{X} (t_k \mid t_{k-1}) (G (t_k) \Delta \underline{X} (t_k \mid t_{k-1}) + \underline{n})^T]$

$= \pi (t_k) G^T (t_k)$

and we have finally

$$W (t_k) [G (t_k) \pi (t_k) G^T (t_k) + N] = \pi (t_k) G^T (t_k)$$

or

$$W (t_k) = \pi (t_k) G^T (t_k) [G (t_k) \pi (t_k) G^T (t_k) + N]^{-1}$$

c. Minimum Variance Differential Correction Applied to Perturbed Dynamic Systems

In Section b, a method for correcting the system coordinates was derived. This method involved the computation of the optimal estimator W. We have assumed that our system consists of an exactly solvable part, which is disturbed by a small perturbation. For this type system, the coordinates at a given time constitute a set of elements for the exactly solvable part. Thus we can consider correcting system elements rather than system coordinates. This approach has two distinct advantages.

First, one can usually determine an analytic form for the state transition matrix Φ for the elements. This is true because we presumably have an exact analytic solution. Notice that using the exact transition matrix for the perturbation transition is equivalent to ignoring second order error terms. This replacement can be made even more consistent by adjusting the exact transition matrix during rectification.

Second, a judicious choice of elements can result in a well-behaved transition matrix. When using the coordinates as elements, the transition matrix is a very rapidly varying function of time. A general criterion for good behavior is that only one element should depend on the total energy. This is discussed more thoroughly in a report from the Goddard Space Flight Center.[1]

The theory of section B can be reformulated in terms of system elements quite easily. Defining the following matrices

$$d\underline{x} = L\, d\underline{\lambda} \qquad d\underline{y} = G\, d\underline{x} = (GL)\, d\lambda = U\, d\lambda$$

$$\Lambda(t) = \Phi(t, t_o)\ \Lambda(t_o)\ \Phi^T(t, t_o) \quad , \pi = L \Lambda L^T$$

$$Y = U \Lambda U^T + N \quad , \quad d\underline{\lambda} = W d\underline{y} \quad , \quad W = \Lambda U^T [\, U \Lambda U^T + N]^{-1}$$

[1] "Goddard Minimum Variance Orbit Determination Program", Goddard Space Flight Center, Greenbelt, Md., October 18, 1962.

$$\Lambda^{+} = \Lambda - WU \quad , Y^{+} = U \Lambda^{+} U^{T}$$

Here we have introduced (matrix)$^{+}$, this is just the new matrix produced by both the transition and optimization processes. If no new information is obtained, (i.e., W = 0) then (matrix)$^{+}$ = (matrix).

Let us now examine the specifics of applying the above results to the parameter optimization problem. If we replace the differentials by derivatives, we can write the transition matrix as

$$\Phi (t, t_o) = \frac{\partial (\underline{x} (t))}{\partial (\underline{x} (t_o))}$$

or in component form

$$\Phi_{ij} (t, t_o) = \frac{\partial x_i (t)}{\partial x_j (t_o)}$$

As an example, consider the problem of determining the length of a rod. Let the exact length be unity, and we know initially that the length is .95 $\pm$ .2 . We have

$$\Phi = 1 \qquad \Lambda_o = .04$$

$$G = U = L = 1$$

with the rest of the results given in Table 1 . Notice that W approaches zero, and thus less importance is placed on later measurements, i.e., the optimal estimator remembers where it is.

t	y	N	dy	W	dy	y	
1	.920	.0064	-.030	.860	-.026	.924	.0056
2	1.030	.0004	+.106	.860	+.091	1.015	.0005
3	1.010	.0001	-.005	.830	-.004	1.011	.0001
4	.970	.0009	-.041	.100	-.004	1.007	.0000
5	1.070	.0049	+.063	.001	.000	1.007	.0000

TABLE I

MINIMUM VARIANCE EXAMPLE

From the discussion in part C of section B, we know that for the two body problem, the coordinates at some time t_i can be written as linear combinations of the coordinates at time t_o. In matrix notation, we have

$$\underline{X}(t_i) = \begin{pmatrix} (P) & (Q) \\ \cdot & \cdot \\ (P) & (Q) \end{pmatrix} X(t_o) = M\,\underline{X}(t_o)$$

where $(\underline{P}) = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$, etc., and the subelements were given explicitly in part C of section I. In component form, we have

$$X_i(t_1) = M_i^{\ j}\, X_j(t_o)$$

and our transition matrix becomes

$$\phi_{ij}(t_i, t_j) = \frac{\partial X_i(t_1)}{\partial X_j(t_o)} = M_i^{\ k}\,\delta_{kj} + \frac{\partial M_i^{\ k}}{\partial X_i}\, X_k$$

The computation of the matrix $\dfrac{\partial M_i^{\ k}}{\partial X_i}$ involves only the computation of the 24 partial derivatives,

$$\frac{\partial P}{\partial X_i}, \frac{\partial Q}{\partial X_i}, \frac{\partial \dot{P}}{\partial X_i}, \frac{\partial \dot{Q}}{\partial X_i}$$

These partial derivatives can be evaluated from the analytic formulas given in part c of section B . The resulting formulas in chain rule form are

$$\frac{\partial P}{\partial X_i} = \frac{a\,(1-\cos\theta)}{r_o} \left(\frac{1}{r_o} \frac{\partial r_o}{\partial X_i} - \frac{1}{a} \frac{\partial a}{\partial X_i} \right.$$

$$= \frac{a}{r_o} \sin\theta \frac{\partial \theta}{\partial X_i}$$

$$\frac{\partial Q}{\partial X} = \sqrt{\frac{a}{k}}\, r_o \sin\theta \left(\frac{1}{r_o} \frac{\partial r_o}{\partial X_i} + \frac{1}{2a} \frac{\partial a}{\partial X_o} \right.$$

$$+ \frac{a d_o}{k} (1 - \cos\theta) \left(\frac{1}{d_o} \frac{\partial d_o}{\partial X_i} + \frac{1}{a} \frac{\partial a}{\partial X_i} \right.$$

$$+ \left(\sqrt{\frac{a}{k}}\, r_o \cos\theta + \frac{a d_o}{k} \sin\theta \right) \frac{\partial \theta}{\partial X_i}$$

$$\frac{\partial \dot{P}}{\partial X_i} = \sqrt{\frac{\mu a}{r r_o}} \left\{ \sin\theta \left[-\frac{1}{2\omega} \frac{\partial a}{\partial X_i} + \frac{1}{r} \frac{\partial r}{\partial X_i} + \frac{1}{r_o} \frac{\partial r_o}{\partial X_o} \right. \right. - \cos\theta \left. \left. \frac{\partial \theta}{\partial X_i} \right] \right\}$$

$$\frac{\partial \dot{Q}}{\partial X_i} = \frac{a}{r} \left[(1 - \cos\theta) \left(\frac{1}{r} \frac{\partial r}{\partial X_o} - \frac{1}{a} \frac{\partial a}{\partial X_i} \right) - \sin\theta \frac{\partial \theta}{\partial X_i} \right]$$

$$\frac{1}{a} \frac{\partial a}{\partial X_i} = 2a \left[\frac{1}{r_o^2} \frac{\partial r_o}{\partial X_i} + \frac{V_o}{k} \frac{\partial V_o}{\partial X_i} \right]$$

$$\frac{\partial \theta}{\partial X_i} = \frac{N}{D}$$

$$N = \left[\left(-\frac{3}{2} \mu (t - t_o) \frac{1}{a} \frac{\partial a}{\partial X_i} - \sin\theta \frac{\partial r_o/a}{\partial X_o} \right.\right.$$

$$\left.\left. -\frac{1}{\sqrt{ak}} (1 - \cos\theta) \left(\frac{\partial d_o}{\partial X_i} - \frac{1}{2a} d_o \frac{\partial a}{\partial X} \right) \right) \right]$$

$$D = \left[1 - \left(\frac{r_o}{a} - 1 \right) \cos\theta + \frac{d_o}{ak} \sin\theta \right]$$

$$\frac{a}{r} \frac{\partial (r/a)}{\partial X_i} = a \left(\cos\theta \frac{\partial (r_o/a)}{\partial X_i} + \frac{\sin\theta}{ak} \left(\frac{\partial d_o}{\partial X_i} - \frac{1 d_o}{2a} \frac{\partial a}{\partial X_i} \right) \right.$$

$$\left. + \frac{1}{r} \left(1 - r_o/a \right) \sin\theta + \frac{d_o}{ak} \cos\theta \right) \frac{\partial \theta}{\partial X_i} \Bigg)$$

The remaining partials (r_o, V_o, d_o) are trivial and will not be recorded here.

In applying these results to any actual problem, the specific types of measurements must be defined. For the numerical calculations the two measurements of range and range rate were arbitrarily selected. The station coordinates are given by

$$X_s = R \cos\theta_1 \cos\left(\Omega (t - t_o) + \theta_3 - \theta_2 \right)$$

$$Y_s = R \cos\theta_1 \sin\left(\Omega (t - t_o) + \theta_3 - \theta_2 \right)$$

$$Z_s = R \sin\theta_1$$

where R is the radius of the earth; Ω, the earth's rotation; θ_1, the station latitude; θ_2, the station longitude; θ_3, the longitude of the vernal equinox at the initial time.

Using these relations we have

$$\text{Range} = \rho = \left((X - X_s)^2 + (y - y_s)^2 + (z - z_s)^2 \right)^{1/2}$$

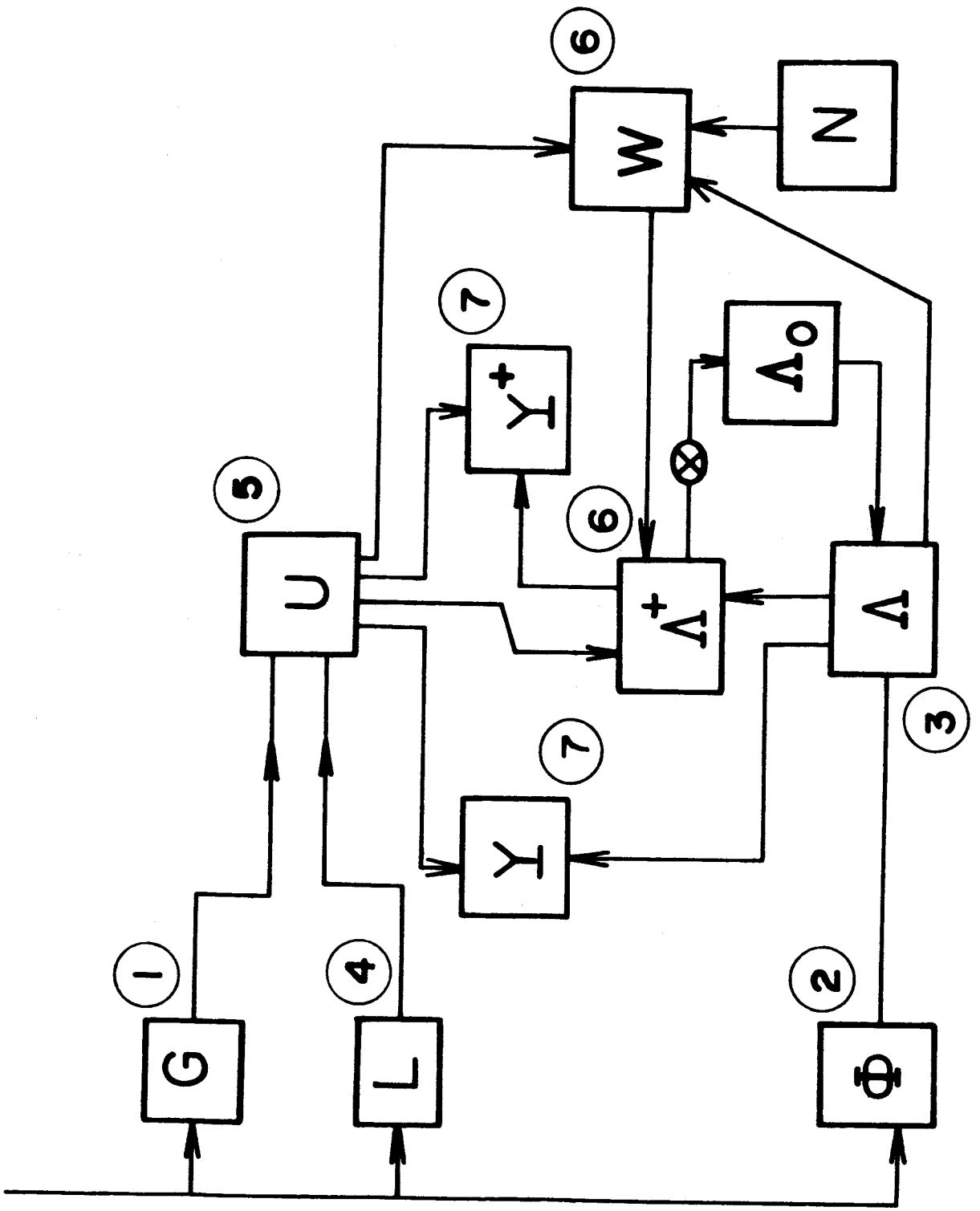


Figure 2
Minimum Variance Block Diagram

$$\text{Range rate} = \dot{\rho} = \left[(X - X_s)(\dot{X} + \Omega y_s) + (y - y_s)(\dot{y} - \Omega x_s) + (z - z_s)\dot{z} \right]$$

Differentiating these expressions, we have the G matrix explicitly.

$$\begin{pmatrix} \frac{1}{\rho}(x - x_s) & \frac{1}{\rho}(y - y_s) & \frac{1}{\rho}(z - z_s) & 0 & 0 & 0 \\ \frac{1}{\rho}(\dot{x} + \Omega y_s) & \frac{1}{\rho}(\dot{y} - \Omega x_s) & \frac{1}{\rho}\dot{z} & \frac{1}{\rho}(x - x_s) & \frac{1}{\rho}(y - y_s) & \frac{1}{\rho}(z - z_s) \\ -\frac{\dot{\rho}}{\rho^2}(x - x_s) & -\frac{\dot{\rho}}{\rho^2}(y - y_s) & -\frac{\dot{\rho}}{\rho^2}(z - z_s) & & & \end{pmatrix}$$

A block diagram showing the minimum variance calculation is shown in Figure 2 . The blocks refer to matrices and the number tags are the order of computation. The switch, ⊗, between Λ^+ and Λ_o indicates that if the measurement is made, Λ_0 is then replaced by Λ^+ .

D. COMPUTER PROGRAM

The entire problem has been programmed for the UNIVAC 1107. The problem consists of thirty-five subprograms, each of which performs a required computation. At present there are 27 pieces of information input to the program. They are listed below, together with representative values:

1.	Maximum number of Newton-Raphson iterations	4
2.	Maximum number of Kepler iterations	5
3.	Allowable Rectification error	1
4.	Time increment	10^{-3} J.D.
5.	Allowable Kepler equation error	10^{-5}
6.	The number of integration per minimum variance calculation	10
7.	Dimensions of dynamic space	6
8.	Dimensions of observation space	2
9.	Maximum time	
10-12.	Initial position coordinates (x, y, z)	10^{3}
13-15.	Initial velocity coordinates (x,y,z)	1
16-18.	Initial position errors (x,y,z)	10
19-21.	Initial velocity errors (x,y,z)	10^{-2}
22.	Latitude of tracking station	
23.	Longitude of tracking station	
24.	Latitude of Vernal Equinox at injection	
25.	Initial time	
26.	Range measurement error	1.8
27.	Range-rate measurement error	.03

The first nine quantities represent initial data, which defines the particular problem involved. These are printed in the format shown in Fig. 3. The rest of the information is related to the injection of the satellite into orbit

and is printed in the format shown in Figure 4. The two body orbit having the initial conditions is calculated, and pertinent information is output in the format shown in Figure 5. Following each integration, the information shown in Figure 6 is printed, and following each minimum variance calculation the information shown in Figure 7 is printed. The formats in Figures 6 and 7 are self explanatory.

The present computer program is quite flexible, and very slight modifications are necessary to include different observations and/or perturbations. However, the method of computing the transition matrix is just numerical evaluation of the chain role differentiation. This computation could be improved by analytically simplifying the partial derivatives.

Initial Data

Time Increment	.100-02 JD
Allowable Kepler Error	.100-06
Maximum Kepler Iterations	5
Dimensions of Dynamic Space	6
Allowable Perturbation Error	.100-06
Maximum Time for Integrations	2438123.100 JD
Dimensions of Observation Space	2
Maximum Newton-Raphson Iterations	5

A Minimum Variance Calculation is Made Every 10 Integrations

Figure 3

Injection Data

Range Error	.31623-01
Injection Time	2438123.000 JD
Range-Rate Error	.31623-01
Station Latitude	.00000
Station Longitude	.00000
Vernal Equinox Longitude	.00000

	X	Y	Z
Injection Coords	.70000+04	.00000	.00000
Injection Errors	.10000+00	.10000+00	.10000+00

	DX/DT	DY/DT	DZ/DT
Injection Coords	.00000	.53033+01	.53033+01
Injection Errors	.10000+00	.10000+00	.10000+00

Figure 4

Two Body Data

Mean Motion	.10977-02
Initial Time	2438123.000 JD
Eccentricity	.12171-01
Initial Speed	.75000+01
Initial Radius	.70000+04
Semi-Major Axis	.69158+04
Position Dot Velocity	.00000

Initial Two Body

X = .70000+04

Y = .00000

Z = .00000

DX / DT = .00000

DY /DT = .53033+01

DZ /DT = .53033+01

Figure 5

Orbit Data

Time ..... 2438123.002 JD

Two Body Eccentric Anomaly 18731-00

Number of Kepler Iterations 1

Metric Between Last Two Iterates ..... .00000

	X	Y	Z
Two Body Coords	.68790+04	.91059+03	.91059+03
Orbit Coords	.68787+04	.91111+03	.91110+03

	DX/DT	DY/DT	DZ/DT
Two Body Coords	-.13970+01	.52116+01	.52116+01
Orbit Coords	-.13997+01	.52114+01	.52112+01

Figure 6

Minimum Variance Data

Range ..... .57383+04

Range Rate ..... .61355+01

Optimization Parameter 84262-00

PHI

.17946+01	.22857-00	.22857-00	.10452+04	.97069+02	.97069+02
.29429-00	.67716-00	.80911-01	.10680+03	.78641+03	.42850+02
.29429-00	.80911-01	.67716-00	.10680+03	.42850+02	.78641+03
.16438-02	.67669-03	.67669-03	.14874+01	.39437-00	.39437-00
.10401-02	-.51946-03	.34890-03	.46052-00	.83115-00	.23689-00
.10401-02	.34890-03	-.51946-03	.46052-00	.23689-00	.83115-00

L

.10000+01	.00000	.00000	.00000	.00000	.00000
.00000	.100000+01	.00000	.00000	.00000	.00000
.00000	.00000	.10000+01	.00000	.00000	.00000
.00000	.00000	.00000	.10000+01	.00000	.00000
.00000	.00000	.00000	.00000	.10000+01	.00000
.00000	.00000	.00000	.00000	.00000	.10000+01

Figure 7a

LAMBDA

.11114+05	.19213+04	.19213+04	.16313+02	.58504+01	.58504+01
.19213+04	.63168+04	.78801+03	.48589+01	.71295+01	.27109+01
.19213+04	.78801+03	.63168+04	.48589+01	.27109+01	.71295+01
.16313+02	.48589+01	.48589+01	.25235-01	.11062-01	.11062-01
.58504+01	.71295+01	.27109+01	.11062-01	.95901-02	.60587-02
.58504+01	.27109+01	.71295+01	.11062-01	.60587-02	.95901-02

LAMBDA PLUS

.31611+04	.86741+03	.98289+03	.48796+01	.20775+01	.21698+01
.86741+03	.32972+04	-.24793+04	.14372+01	.30793+01	-.15374+01
.98289+03	-.24793+04	.27759+04	.14290+01	-.16089+01	.25912+01
.48796+01	.14372+01	.14290+01	.75356-02	.32875-02	.32810-02
.20775+01	.30793+01	-.16089+01	.32875-02	.34236-02	-.32329-03
.21698+01	-.15374+01	.25912+01	.32810-02	-.32329-03	.29763-02

G TRANSPOSE

-.38305-00	-.64602-03	-.38305-00	-.64602-03	.10068+01	-.14887+04
.61742-00	-.19258-03	.61742-00	-.19258-03	.10269+01	-.43699+03
.68707-00	-.18711-03	.68707-00	-.18711-03	.10939+01	-.43728+03
.00000	-.38305-00	.00000	-.38305-00	.20390-02	-.22989+01
.00000	.61742-00	.00000	.61742-00	.15791-02	-.10018+01
.00000	.68707-00	.00000	.68707-00	.16327-02	-.10020+01

Y

.57692+04	.50779+01
.50779+01	.68181-02

Y PLUS

.98383-03	-.59605-07
.93130-06	.70137-03

Figure 7b

III. METHOD OF SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

The following presentation deals with an effective method of solving a set of nonlinear differential equations subject to the knowledge of a complete set of boundary values distributed between two points. This presentation is hoped to be a more lucid yet more general explanation of the method presented by McGill and Kenneth.[1]

Let us consider the set of nonlinear differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_n, t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, \ldots, x_n, t) \end{pmatrix} \tag{1}$$

Further, we will use the Taylor's series for functions of several variables to obtain

$$f_i(x_1, \ldots, x_n, t) = f_i(s_1, \ldots, s_n, t) + \frac{\partial f_1}{\partial s_1}(x_1 - s_1)$$

$$+ \ldots + \frac{\partial f_1}{\partial s_n}(x_n - s_n) + O\left[(x_i - s_i)\right]^2$$

Should we assume that our x_i lie "sufficiently near" the s_i, we could approximate $f_i(X, t)$ by this truncated Taylor's Series and thus obtain

$$f_1(x_1, \ldots, x_n, t) = f_1(s_1, \ldots, s_n, t) + \frac{\partial f_1}{\partial s_1}(x_1 - s_1) + \ldots + \frac{\partial f_1}{\partial s_n}(x_n - s_n)$$

$$\cdot \qquad \cdot$$

$$\cdot \qquad \cdot$$

$$f_n(x_1, \ldots, x_n, t) = f_n(s_1, \ldots, s_n, t) + \frac{\partial f_n}{\partial s_1}(x_1 - s_1) + \ldots + \frac{\partial f_n}{\partial s_n}(x_n - s_n)$$

---

[1] Solution of variational problems by means of a generalized Newton-Raphson operator. Research Department, Grumman Aircraft Engineering Corporation; Bethpage, New York

$$\begin{pmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} f_1(s_1, \ldots, s_n, t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(s_1, \ldots, s_n, t) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial s_1} & \cdots & \frac{\partial f_1}{\partial s_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n}{\partial s_1} & \cdots & \frac{\partial f_n}{\partial s_n} \end{pmatrix} \cdot \begin{pmatrix} x_1 - s_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - s_n \end{pmatrix}$$

For convenience we denote the matrix by $J(S, t)$, the Jacobian, the vectors

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix} \text{ and } \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \quad \text{by} \quad \theta, S, \text{ and } F \text{ respectively.}$$

In this new notation our system of truncated equations become

$$\dot{\theta} = F(S, t) + J(S, t)\,[\,\theta - S\,] \qquad (2)$$

Since this is a linear system in θ, it is obvious that were S given and the functional form of F and J known, we could, with a complete set of initial values at $t = t_o$, integrate this system. Of course, if one had a complete system of initial values at $t = t_o$, one could have solved the very first system of equations, Equation (1), numerically in a perfectly routine manner.

We wish, however, to solve the two point boundary value problem in which there is given only a partial set of k values at $t = t_o$

$$x_1(t_o) = x_{10},$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$x_k(t_o) = x_{ko} : k < n.$$

At a subsequent time $t = t_1$ we have an additional set of $n - k$ values:

$$x_m(t_1) = x_{m1}$$

$$\vdots$$

$$x_p(t_1) = x_{p1} \qquad \text{where we have } n - k \text{ identities.}$$

We select a first guess $\theta^1 = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ such that a total of n boundary values is satisfied, and, in general, make no further demands on θ^1 except that it be n times differentiable. So now,

$$\dot{\theta}^2 = F(\theta^1, t) + J(\theta^1, t) \quad [\theta^2 - \theta^1] \qquad (3)$$

is a linear system in θ^2, due to elimination of higher order terms. We are able to generate, at least numerically, a new vector θ^2 which is based on the original differential system and which will be required to satisfy the given boundary conditions. This method is definitely better than the so called "shotgun" method, in which the $n - k$ unknown initial values are treated as parameters and varied about initial guesses. For each variation, the nonlinear system is numerically integrated, and from the entire spectrum of solutions the one satisfying most nearly the desired end point boundary values is selected. The linearized approach used here allows us to use the unique linear properties of the particular integral and complimentary solutions to construct a solution to the boundary value problem, in a simple, algebraic manner.

First we generate numerically a solution U_p which satisfies Equation (3) subject to the partial set of initial values.

$$U_p(t_o) = \begin{pmatrix} x_{10} \\ \vdots \\ x_{k0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We compute U_p from $t = t_o$ to $t = t_1$.

We will consider next the equation homogeneous to Equation (3) which is

$$\dot{X} - J(\theta^1, t)\, X = 0 \tag{4}$$

and solve it subject to the initial conditions

$X(t = t_o) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, the 1 occurring in the $k + 1^{st}$ row, i.e. the first variable unknown at time t_o.

We call this complementary solution $U_{\text{complementary 1}}$ U_{c1}. We now solve the same differential equation subject to the initial conditions

$$X(t = t_o) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$, the 1 occurring in the k + 2nd row.

We call the solution to this problem U_{c2}. Continuing in this manner, we obtain a solution to Equation (4) satisfying

$$X\,(t = t_o) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \text{, the 1 occurring in the } n^{th} \text{ row ,}$$

which is called $U_{c(n-k)}$. This gives us a complete set of complementary solutions.

Since we have a linear system of equations,

$$\theta^2 = U_p + A_1 U_{c1} + \cdots + A_{n-k} U_{c(n-k)} \qquad (5)$$

is a solution vector of Equation (3). We now employ the n-k conditions that occur at $t = t_1$ to solve this n-k by n-k system of algebraic equations for the constants $A_1, \ldots, A_{n-k}$. This choice of $A_1, \ldots, A_{n-k}$ forces θ^2 to satisfy the n-k constants at $t = t_1$, and our method of computing $U_p, U_{c1}, \ldots, U_{c(n-k)}$ automatically requires θ^2 to satisfy Equation (3) and the boundary conditions at $t = t_o$. Hence θ^2 is a solution vector satisfying all boundary conditions.

We are now able to repeat this entire process and obtain

$$\dot{\theta}^3 = F(\theta^2, t) + J(\theta^2, t)\,[\theta^3 - \theta^2], \quad \text{etc.}$$

We observe that if :

(1) $||X^m - X^{m-1}|| < \varepsilon$, $t_o \leq t \leq t_1$, where $||X||$ is the norm of X,

(2) $J(X^{m-1}, t)$ is a bounded operator, that is,

$$\left|\left| J\,(X^{m-1}, t)\,(X^m - X^{m-1}) \right|\right| = 0(\varepsilon), \text{ if } \left|\left| X^m - X^{m-1} \right|\right| < \varepsilon,$$

and (3) $\left|\left| F\,(X^{m-1}, t) - F\,(X^m, t) \right|\right| = 0\,(\varepsilon)$ when $\left|\left| X^m - X^{m-1} \right|\right| < \varepsilon$

then $\dot{X}^m = F\,(X^{m-1}, t) + J\,(X^{m-1}, t)\,[\,X^m - X^{m-1}]$, and our general iteration equation becomes

$$\dot{X}^m = F\,(X^m, t) + [\,F\,(X^{m-1}, t) - F\,(X^m, t)\,] + J\,(X^{m-1}, t)\,(X^m - X^{m-1}).$$

So that $\dot{X}^m = F\,(X^m, t) + O\,(\varepsilon)$, that is, X^m is the solution.

This is a simplified set of necessary conditions and may be easily replaced by a more sophisticated set. A paper by McGill and Kenneth, "A convergence Theorem on the Iterative Solution of Nonlinear Two Point Boundary Value Systems", proves that, subject to certain restrictions, the set of vector X^n do converge to the solution vector.

IV. ERRATA SHEET

(For Report No. 1 of "Parameter Optimization")

Page 13, 14th line

then $= 0$

should read

then $\alpha = 0$

Page 15, 6th line

at some point $\underline{v}_1$

should read

at some point $\underline{r}_1$

Page 15, Eq. (3.3)

$$p = \frac{L^2}{M_s M_e G}$$

should read

$$p = \frac{L^2}{M_s^2 M_e G}$$

Page 16, Eq. (3.4)

$$\theta_o = 1 - \cos^{-1} \frac{1}{e} \left(\frac{P}{r_1} - 1 \right)$$

should read

$$\theta_o = \theta_1 - \cos^{-1} \frac{1}{e} \left(\frac{P}{r_1} - 1 \right)$$

Page 17, Eq. (3.7)

$$K \left(\frac{1}{r_1}_2 - \frac{1}{r_2}_2 \right) = \cdots$$

should read

$$K^2 \left(\frac{1}{r_1}_2 - \frac{1}{r_2}_2 \right) =$$

Page 18, 8th line

$r_1 < r_2$ then $\theta_o = \theta$.

should read

$r_1 < r_2$ then $\theta_o = \frac{\pi}{2} + \theta$

Page 18, 9th line

$r_1 > r_2$ then $\theta_o = \theta_1 - \pi$

should read

$r_1 > r_2$ then $\theta_o = \theta_1$

(For Report No. 2 of "Parameter Optimization")

Page 25, 8th equation

$$\left| \underline{L} \right| = \left| M_s r \times \underline{V} \right| = M_s r V \cos \theta = M_s r V \sin \theta$$

should read

$$\left| \underline{L} \right| = \left| M_s \underline{r} \times \underline{V} \right| = M_s r V \cos \theta = M_s r V \sin \phi$$

Page 25, 9th equation

$$\frac{\partial L}{\partial \theta} = M_s r V \cos \theta$$

should read

$$\frac{\partial L}{\partial \theta} = - M_s r V \sin \theta$$

Page 26, 2nd equation

$$\frac{\partial P}{\partial \theta} = \frac{2L}{M_s^2 M_e G} \frac{\partial L}{\partial \theta} = \frac{rV}{K} \cos \theta$$

should read

$$\frac{\partial P}{\partial \theta} = \frac{2L}{M_s^2 M_e G} \frac{\partial L}{\partial \theta} = \frac{- r^2 V^2 \sin 2\theta}{K}$$

Page 26, Last part of Eq. (14)

$$dP = \frac{r V \cos \theta}{K} d\theta$$

should read

$$dP = \frac{-r^2 V^2 \sin 2\theta}{K} d\theta$$

Page 29, Eq. (19)

$$d\theta_o = \frac{1}{e}\sqrt{\frac{P}{K}} \cos(\theta - \theta_o)\, dV_r$$

should read

$$d\theta = \frac{-1}{e}\sqrt{\frac{P}{K}} \cos(\theta - \theta_o)\, dV_r$$

Page 30, 4th equation

$$\frac{\partial P}{\partial \dot{\theta}} = \frac{2L}{M_s M_e G} \frac{\partial L}{\partial \dot{\theta}} = \frac{2L\, r_1^2}{M_s M_e G}$$

should read

$$\frac{\partial P}{\partial \dot{\theta}} = \frac{2L}{M_s^2 M_e G} \frac{\partial L}{\partial \dot{\theta}} = \frac{2L\, r_1^2}{M_s M_e G}$$

Page 30, Eq. (20)

$$dP = \frac{2L\, r_1^2}{M_s M_e G} d\dot{\theta} = \frac{2\, r^4 \dot{\theta}}{M_e G} d\dot{\theta}$$

should read

$$dP = \frac{2Lr_1^2}{M_s M_e G} d\dot{\theta} = \frac{2\, r_1^4 \dot{\theta}_1}{M_e G} d\dot{\theta}$$

Page 32, 5th equation

$$r^2 \frac{Ke}{C} \sin(\theta - \theta_o) + K \sin(\theta - \theta_o)(Pa - r^2) \frac{C}{Kea} = -Ke \cos(\theta - \theta_o) \frac{\partial \theta_o}{\partial \dot{\theta}}$$

should read

$$r^2 \frac{Ke}{C} \sin(\theta - \theta_o) - K \sin(\theta - \theta_o)(Pa - r^2) \frac{C}{Kea} = -Ke \cos(\theta - \theta_o) \frac{\partial \theta_o}{\partial \dot{\theta}}$$

Page 32, Last part of the 6th equation

$$(p^2 - r^2) \frac{r}{Ce} \frac{\sin(\theta - \theta_o)}{(p - r)} = (p + r) \frac{\sin(\theta - \theta_o)}{Ke}$$

should read

$$(p^2 - r^2) \frac{r}{Ce} \frac{\sin(\theta - \theta_o)}{(p - r)} = \frac{(p + r)\, r}{Ce} \sin(\theta - \theta_o)$$

Page 32, Last equation

$$d\theta_o = \frac{(p + r)}{Ke} \sin(\theta - \theta_o)\, d\dot{\theta}$$

should read

$$d\theta_o = \frac{(p + r)\, r}{Ce} \sin(\theta - \theta_o)\, d\dot{\theta}$$

Explanation of Notations:

K stands for $r^2\dot{\theta}$ in report No. 1 and for GM_e in this report. Throughout this report C stands for $r^2 \dot{\theta}$.

(For Report No. 3 of "Parameter Optimization")

Page 5, 3rd equation

$$2C \frac{\partial C}{\partial r} = K (1 + e \cos \theta) + K \cos \theta \frac{\partial e}{\partial r}$$

should read

$$2C \frac{\partial C}{\partial r} = K (1 + e \cos \theta) + K r \cos \theta \frac{\partial e}{\partial r}$$

Page 5, 5th equation

$$K \cos \theta \frac{\partial e}{\partial r} = \frac{4C^2}{r} - \frac{C^2}{r}$$

should read

$$r K \cos \theta \frac{\partial e}{\partial r} = \frac{4C^2}{r} - \frac{C^2}{r}$$

Page 5, 6th equation

$$\frac{\partial e}{\partial r} = \frac{3P}{r \cos \theta} = \frac{3 a (1-e^2)}{r \cos \theta}$$

should read

$$\frac{\partial e}{\partial r} = \frac{3P}{r \cos \theta} = \frac{3 a (1-e^2)}{r^2 \cos \theta}$$

Page 6, 2nd equation

$$(1 - e^2)\ \frac{\partial a}{\partial r} = \frac{4P}{r} + \frac{6\ ae\ P}{r \cos\theta} = \frac{2P\ (2 \cos\theta + 3\ ae)}{r^2 \cos\theta}$$

should read

$$(1 - e^2)\ \frac{\partial a}{\partial r} = \frac{4P}{r} + \frac{6\ ae\ P}{r^2 \cos\theta} = \frac{2P\ (2r \cos\theta + 3\ ae)}{r^2 \cos\theta}$$

Page 6, 3rd equation

$$\frac{\partial a}{\partial r} = \frac{2a\ (2 \cos\theta + 3\ ae)}{r \cos\theta}$$

should read

$$\frac{\partial a}{\partial r} = \frac{2a\ (2r \cos\theta + 3\ ae)}{r^2 \cos\theta}$$

Page 6, 6th equation

$$\frac{3P}{r} - 3P = er \sin(\theta - \theta_o)\ \frac{\partial \theta_o}{\partial r}$$

should read

$$\frac{4P}{r} = \frac{P}{r} + er \sin(\theta - \theta_o)\ \frac{\partial \theta_o}{\partial r} + \frac{3P}{r}$$

Page 6, 7th equation

$$\frac{\partial \theta_o}{\partial r} = \frac{3P}{er \sin(\theta - \theta_o)}\ \left[\ \frac{1}{r} - 1\ \right]$$

should read

$$\frac{\partial \theta_o}{\partial r} = 0$$

Page 7, Eq. (6)

$$\frac{\partial P}{\partial \theta} = \frac{2\, ae\, r \sin\theta}{r\cos\theta + 2\, ae}$$

should read

$$\frac{\partial P}{\partial \theta} = \frac{2\, ae^2\, r \sin\theta}{r\cos\theta + 2\, ae}$$

Page 9, Eq. (13)

$$\frac{\partial P}{\partial r_1} = \frac{1}{M_s^2 M_e G} \frac{\partial L^2}{\partial r_1}$$

$$= \frac{1}{M_s^2 M_e G} \cdot M_s^2 \left\{ 2 M_e G \left(\frac{r_2}{r_2 + r_1} - \frac{r_1 r_2}{(r_2 + r_1)^2} \right) + \right.$$

$$\left. + (\dot{r}_2^2 - \dot{r}_1^2) \left(\frac{2 r_1 r_2^2}{r_2^2 + r_1^2} + \frac{2 r_1^3 r_2^3}{r_2^2 - r_1^2)^2} \right) \right\}$$

$$= \frac{1}{M_e G} \left\{ 2 M_e G \frac{r_2^2}{r_2^2 + r_1^2} + (\dot{r}_2^2 - \dot{r}_1^2) \frac{r_1 r_2^4}{(r_2^2 - r_1^2)^2} \right\}$$

should read

$$\frac{\partial P}{\partial r_1} = \frac{1}{M_s^2 M_e G} \frac{\partial L^2}{\partial r_1}$$

$$= \frac{1}{M_s^2 M_e G} M_s^2 \left\{ 2 M_e G \left(\frac{r_2}{r_2 + r_1} - \frac{r_1 r_2}{(r_2 + r_1)^2} \right) + \right.$$

$$+ (\dot{r}_2^2 - \dot{r}_1^2) \left(\frac{2 r_1 r_2^2}{r_2^2 - r_1^2} + \frac{2 r_1^3 r_2^2}{(r_2^2 - r_1^2)^2} \right) \Bigg\}$$

$$= \frac{2 r_2^2}{(r_2 + r_1)^2} + \frac{2 r_1 r_2^4 (\dot{r}_2^2 - \dot{r}_1^2)}{M_e G (r_2^2 - r_1^2)^2}$$

Page 9, Eq. (14)

$$\frac{\partial L^2}{\partial \dot{r}_1} = \frac{M_s^2}{M_e G} \left\{ 2 M_e G \frac{r_2^2}{r_2^2 + r_1^2} + (\dot{r}_2^2 - \dot{r}_1^2) \frac{r_1 r_2^4}{r_2^2 - r_1^2)^2} \right\}$$

should read

$$\frac{\partial L^2}{\partial \dot{r}_1} = 2 M_s^2 \left\{ \frac{M_e G r_2^2}{(r_2 + r_1)^2} + \frac{r_1 r_2^4 (\dot{r}_2^2 - \dot{r}_1^2)}{(r_2^2 - r_1^2)^2} \right\}$$

Page 10, Eq. (16)

$$\frac{\partial L^2}{\partial \dot{r}_1} = \frac{2 \dot{r}_1}{M_s^2 M_e G} \left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \right)$$

should read

$$\frac{\partial L^2}{\partial \dot{r}_1} = -2 \dot{r}_1 M_s^2 \left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \right)$$

Page 10, 3rd equation

$$e = \sqrt{1 + \frac{2\,EL^3}{M_s^2 M_e^2 G^2}}$$

should read

$$e = \sqrt{1 + \frac{2\,EL^2}{M_s^2 M_e^2 G^2}}$$

Page 10, Last equation

$$\frac{2L^2}{M_s^2 M_e^2 G^2}\left[\frac{1}{2M_s}\left(\frac{\partial L^2}{\partial r_1} - \frac{\partial L^2}{r_1^3}\right) + \frac{G M_e M_s}{r_1^2}\right] L^2$$

should read

$$\frac{2L^2}{M_s^2 M_e^2 G^2}\left[\frac{1}{2 M_s^3}\left(\frac{1}{r_1^2}\,\frac{\partial L^2}{\partial r_1} - \frac{2L^2}{r_1^3}\right) + \frac{GM_e M_s}{r_1^2}\right]$$

Page 11, Last part of Eq. (17)

$$= \frac{2}{M_s^2 K^2}\left\{\frac{1}{2M_s}\left[\frac{K}{r_1^2} - \frac{2L^2}{r_1^3}\right] L^2 + \left[\frac{M_s}{2}\left(\dot{r}_1^2 + \frac{2L^2}{M_s^2 r_1^2}\right) - \frac{M_s K}{r_1}\right]\frac{\partial L^2}{\partial r_1}\right\}$$

should read

$$= \frac{2}{M_s^2 K^2}\left\{\frac{1}{2M_s}\left[\frac{K}{r_1^2} - \frac{2L^2}{M_s^2 r_1^3}\right] L^2 + \left[\frac{M_s}{2}\left(\dot{r}_1^2 + \frac{L^2}{M_s^2 r_1^2} + \frac{L^2}{M_s^4 r_1^2}\right) - \frac{M_s K}{r_1}\right]\frac{\partial L^2}{\partial r_1}\right\}$$

Page 11, Last part of Eq. (18)

$$= \frac{2}{M_s K^2}\left\{\dot{r}_1 L^2 + M_s\left[\frac{\dot{r}_1^2}{2} + \frac{L^2}{M_s^2 r_1^2} - \frac{K}{r_1}\frac{\partial L^2}{\partial \dot{r}_1}\right]\right\}$$

should read

$$= \frac{2}{M_s K^2}\left\{\dot{r}_1 L^2 + M_s\left[\frac{\dot{r}_1^2}{2M_s} + \frac{L^2}{M_s^3 r_1^2} - \frac{K}{r_1}\right]\frac{\partial L^2}{\partial \dot{r}_1}\right\}$$

Page 19, 3rd line

$$\frac{\xi^2 \cos^2\theta}{4} - + 1 > 0$$

should read

$$\frac{\xi^2 \cos^2\theta}{4} - \xi + 1 > 0$$